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Continuous phase transition in a disordered eight-states Potts model

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Abstract. We investigate the two-dimensional eight-states ferromagnetic Potts model in the Voronoi-Delaunay tessellation. In this study, we assume that the coupling factor J varies with the distance r between the first neighbors as $J(r) \propto \mathrm{e}^{-\alpha r}$, with $\alpha \geq 0$. The disordered system is simulated applying the single-cluster Monte-Carlo update algorithm and the reweighting technique. We find that this model displays a first-order phase transition if $\alpha = 0.0$, in agreement with previous recent studies. For $\alpha = 0.5$ and 1.0, a typical second order transition is observed and the critical exponents for magnetization and susceptibility are calculated.

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1 Introduction

It is well-known that the ferromagnetic Potts model with q-states in two-dimensional lattices displays first order phase transitions for $q > 4$ [1,2]. In a recent paper, Chen *et al.* [3] studied the effect of quenched, bond randomness on the nature of the phase transition in the two-dimensional eight-states Potts model. Their model is a restricted version of the random bond Potts model discussed by Wu [1] with two ferromagnetic couplings factors, J_1 and J_2, chosen with probabilities p and $(1-p)$, respectively. Using Monte-Carlo simulations, they showed that, at sufficiently high values of J_1/J_2, the transition changes from first to second order. Janke *et al.* [4] used the Voronoi-Delaunay tessellation in two-dimensions to study the ferromagnetic Ising model. They found that the critical exponents agree with the exactly known critical exponents for regular lattices. In a more recent paper, Janke and Villanova [5] have used the same lattice for the eight-states Potts model to demonstrate that this kind of quenched bond disorder (disorder in the number of coordination, not in the coupling factor J) does not change the order of transition. Recently, Berche *et al.* [6] performed a Monte-Carlo simulation in a two-dimensional eight-states ferromagnetic Potts model with an aperiodic modulation of the exchange coupling. They found that, depending on the kind of modulation, the critical exponents assume different values.

In the present study, we investigate the same model of Janke and Villanova [5], and use an entirely similar algorithm to compute the relevant thermodynamical properties of the system. However, considering that the distance r between first neighbors changes randomly from neighbor to neighbor, here we assume that the coupling factor J varies exponentially with r. In the limiting case of constant J, we recover the results of Janke and Villanova [5]. In the general case of $J(r) \propto \mathrm{e}^{-\alpha r}$, we show that, depending on the value of α, the magnetic phase transition changes from first to second order.

2 Model and simulation

We consider the eight-states Potts ferromagnet ($q = 8$) in a Poissonian random lattice where the coordination number varies locally. The Voronoi construction or tessellation for a given set of points in the plane is defined as follows. For each point, we first determine the polygonal cell consisting of the region of space nearer to that point than to any other point. Whenever two such cells share an edge, they are considered as neighbors. From the Voronoi tessellation, we can obtain the dual lattice by the following procedure. When two cells are neighbors, one draws a link between the two points located in the cells. From the links, one obtains the triangulation of space that is called the Delaunay random lattice. The Delaunay lattice is dual to the Voronoi tessellation in the sense that points correspond to cells, links to edges and triangles to the vertices of the Voronoi tessellation.

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The Hamiltonian of an eight-states ferromagnetic Potts model can be written as

$$-KH = \sum_{<i,j>} J_{ij}\delta_{\sigma_i\sigma_j}, \quad (1)$$

where $K = 1/k_{\rm B}T$, T is the temperature, $k_{\rm B}$ is the Boltzmann constant, δ is the Kronecker delta function, the sum goes over all nearest-neighbors pairs of sites (points in the Voronoi construction) and the spin σ can take the values $\sigma = 1, ..., 8$. Here, we assume that the coupling factor J_{ij} depends on the relative distance r_{ij} between sites i and j according to the following expression:

$$J_{ij} = J_0 {\rm e}^{-\alpha r_{ij}}, \quad (2)$$

where J_0 is a constant and $\alpha \geq 0$ is a model parameter.

We apply the single-cluster update algorithm [7] to perform extensive simulations of the model. For $\alpha = 0.0$, we use 20 realizations of different lattice sizes comprising a number $N = 250, 500, 1000, 2000$ and 4000 of sites near the respective transition points $K = 0.826, 0.830, 0.830, 0.832$, and 0.833. After thermalization, we record 10^6 measurements (taken after the flip of 1, 1, 1, 2 or 4 clusters, depending on the value of K, respectively) of the energy E and magnetization $M = (q\max[n_i] - N)/(q-1)$ in a time series file, where $n_i \leq N$ denotes the number of spins with "orientation" $i = 1, ..., 8$ in one lattice configuration. Obviously, it is sufficient to store the integer $N/q \leq \max[n_i] \leq N$. We follow the same procedure for simulations with $\alpha > 0$. In this case, however, 72 independent replicas are generated for each lattice size with 500, 1000, 2000, 4000 and 8000 sites. From these data, we compute all the quantities of interest as a function of temperature by means of reweighting techniques [8]. For instance, the specific heat $C(K)$ is calculated as

$$C(K) = [C^{(i)}(K)] \equiv (1/R)\Sigma_i^R C^{(i)}(K), \quad (3)$$

where $C^{(i)}(K) = K^2 N(< e^2 > - < e >^2)$ corresponds to the replica labeled by superindex (i), e is the free energy E normalized by the number of sites N, the square brackets denote the replica average and the variable R represents the number of replicas used in the simulations. The replica average is computed over $C^{(i)}$ and not over the energy moments because quenched averages should be calculated at the level of the free energy and not the partition function [9]. Finally, we determine the maximum value, $C_{\max} = C(K_{C_{\max}})$, for each lattice size and study the finite size scaling behavior of $C_{\max}$ and $K_{C_{\max}}$. Other thermodynamics quantities of the system have been accessed and analyzed along the same lines.

3 Results and conclusions

Our simulations with $\alpha = 0.0$ indicate that the model displays a first order phase transition, in perfect agreement with the results reported by Janke and Villanova [5]. However, at sufficiently high values of α (*e.g.*, $\alpha = 0.5$ and $\alpha = 1.0$), we observe a typical second order transition. In order to verify the order of these transitions, we apply finite-size scaling (FSS) [10] in conjunction with histogram techniques. Initially we search for the minima of the energetic fourth-order parameter

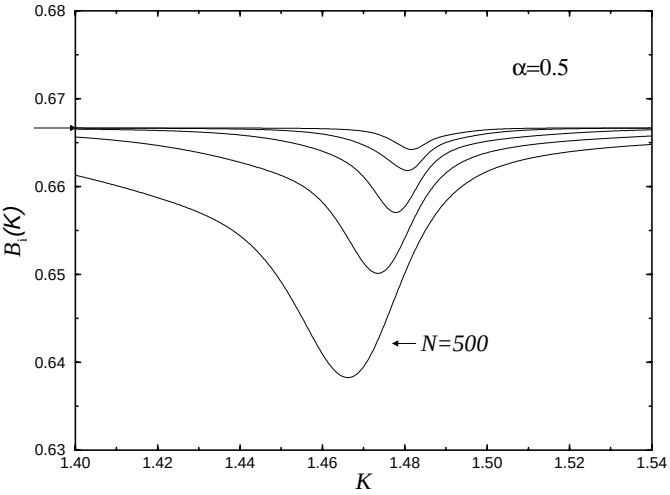

Fig. 1. Plot of the Binder parameter $B_i(K)$ *versus* K for $\alpha = 0.5$ and several lattices sizes ($N = 500, 1000, 2000, 4000$ and 8000 sites). The arrow indicates the position $B_i(K) = 2/3$.

$$B_i(K) = 1 - \frac{< e^4 >}{3 < e^2 >^2} \cdot \quad (4)$$

This quantity, also known as the *Binder parameter*, gives a qualitative as well as a quantitative description of the order of the transition [11]. It is known [12] that this parameter takes a minimum value $B_{i,\min}$ at the effective transition temperature $T_{\rm c}(N)$. One can show[13] that for a second-order transition $\lim_{N\to\infty}(2/3 - B_{i,\min}) = 0$, even at $T_{\rm c}$, while at a first-order transition the same limit measures the latent heat $|e_+ - e_-|$:

$$\lim_{N\to\infty}\left(\frac{2}{3} - B_{i,\min}\right) = \frac{1}{3}\frac{(e_+ - e_-)^2(e_+ + e_-)^2}{(e_+^2 + e_-^2)^2} \cdot \quad (5)$$

In Figure 1, we exhibit the plot of $B_i(K)$ *versus* K for $\alpha = 0.5$ and different lattice sizes (from $N = 500$ to 8000 sites). We can see that, in the limit of large lattice sizes, the Binder parameter goes asymptotically to 2/3, providing a qualitative confirmation for the presence of a continuous transition in the system. For $\alpha = 0.0$ (see Fig. 2), however, the Binder parameter clearly goes to a value which is different of 2/3. This is a sufficient condition to characterize a first-order transition. The order of the transitions can be confirmed by plotting the values of $2/3 - B_{i_{\min}}(K)$ against $1/N$ for different values of α. While for $\alpha = 0.5$ and 1.0 the curve goes to zero as we increase the system size (Fig. 3), for $\alpha = 0.0$ the quantity $2/3 - B_{i_{\min}}(K)$ approaches a nonvanishing value in the limit of small $1/N$ (Fig. 4). At this point, we can assume that a change in α from 0.0 to 0.5, should be followed by a crossover at a value $\alpha = \alpha_{\rm c}$ from a first order to a continuous transition. Additional simulations are necessary to determine the precise value of $\alpha_{\rm c}$.

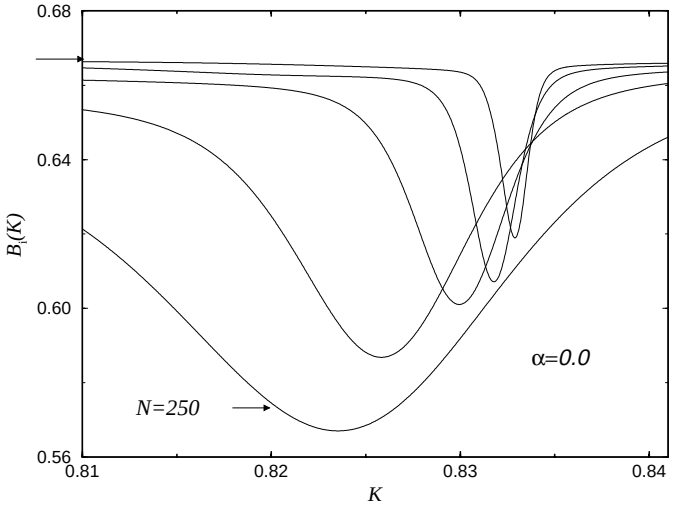

Fig. 2. The same as in Figure 1, but for $\alpha = 0.0$.

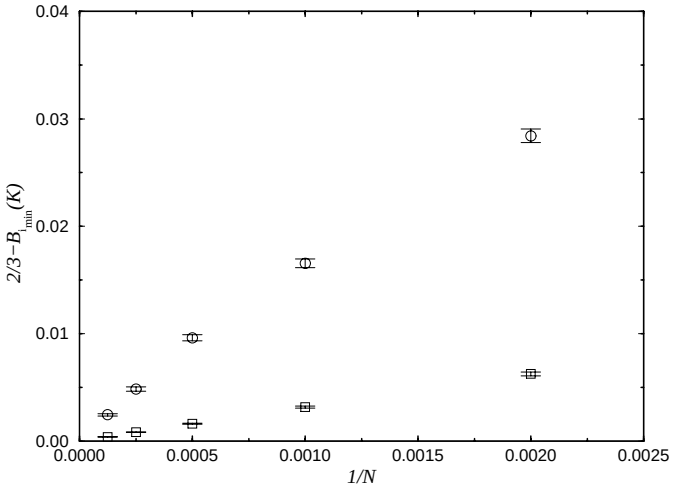

Fig. 3. Plot of $2/3 - B_{i_{\min}}(K)$ *versus* $1/N$ for $\alpha = 0.5$ (circles) and 1.0 (squares).

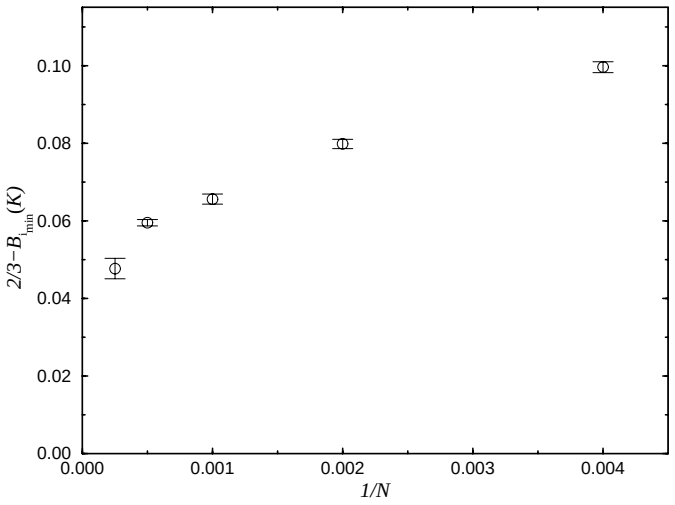

Fig. 4. Plot of $2/3 - B_{i_{\min}}(K)$ *versus* $1/N$ for $\alpha = 0.0$.

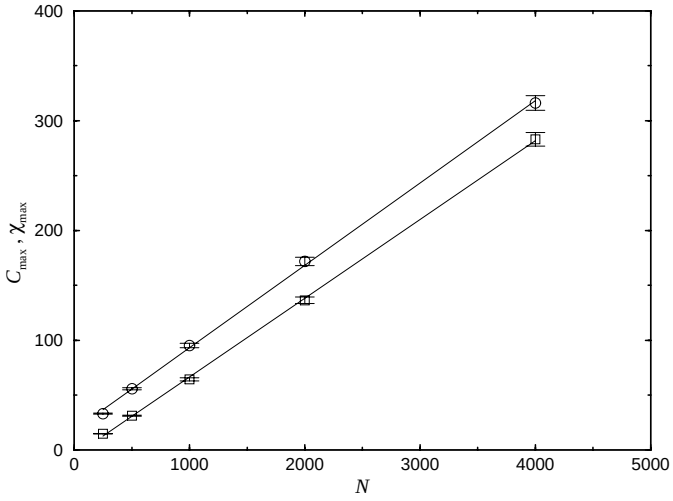

Fig. 5. Plot of the specific heat $C_{\max}$ (circles) and susceptibility $\chi_{\max}$ (squares) *versus* N for $\alpha = 0.0$.

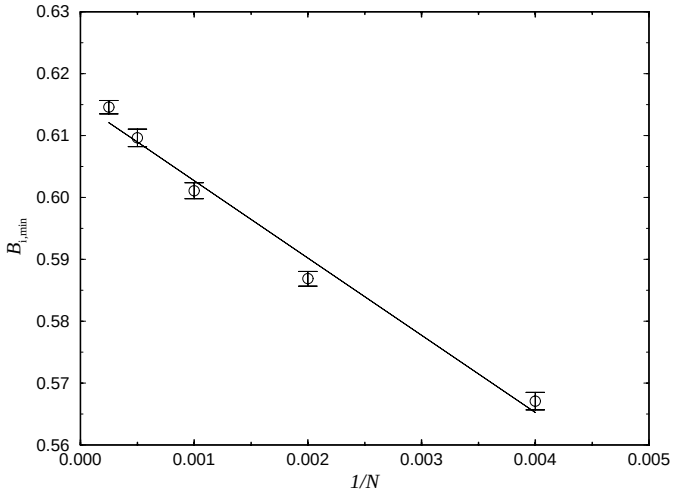

Fig. 6. Plot of the Binder parameter $B_{i,\min}$ *versus* $1/N$ for $\alpha = 0.0$.

A more quantitative analysis can be carried out through the FSS of the specific heat $C_{\max}$, the susceptibility maxima $\chi_{\max}$ and the minima of the Binder parameter $B_{i,\min}$. If the hypothesis of a first-order phase transition for $\alpha = 0.0$ is correct, we should then expect, for large system sizes, an asymptotic FSS behavior of the form [5,9],

$$C_{\max} = a_C + b_C N + ... \quad (6)$$

$$\chi_{\max} = a_\chi + b_\chi N + ... \quad (7)$$

$$B_{i,\min} = a_{B_i} + b_{B_i}/N + ... \quad (8)$$

As depicted in Figure 5, our results for the scaling of the specific heat and susceptibility are consistent with equations (6, 7). In Figure 6 we show the scaling of the Binder parameter minima, and again the first order phase transition is verified. Next, we use the reweighting technique and FSS to calculate the magnetization fourth-order Binder parameter and estimate the critical temperatures for $\alpha = 0.5$ and 1.0. For the largest lattice size, we obtain $K_c \approx 1.484$ and 2.584, respectively.

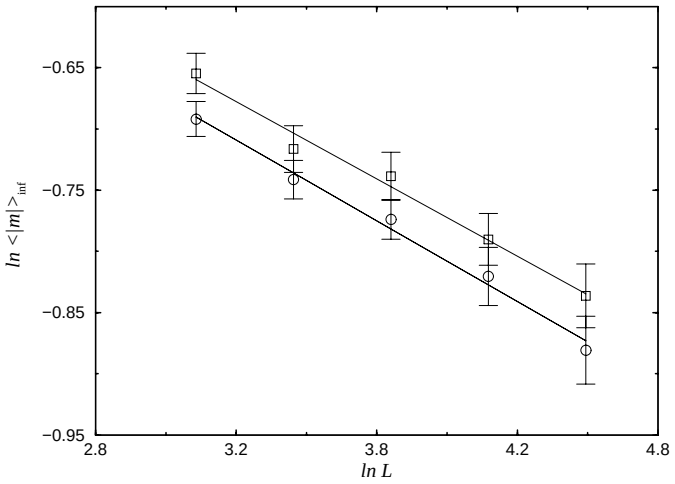

Fig. 7. Logarithmic plot of the magnetization at the inflection point *versus* the size system L for $\alpha = 0.5$ (circles) and $\alpha = 1.0$ (squares).

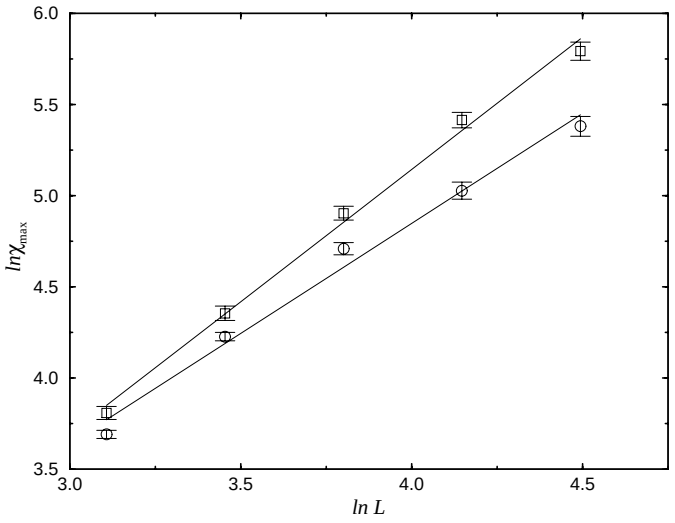

Fig. 8. Logarithmic plot of the maximum susceptibility $\chi_{\max}$ *versus* the size system L for $\alpha = 0.5$ (circles) and 1.0 (squares).

In order to estimate the exponent ratios β/ν and γ/ν, here we use the fact that the magnetization at the inflection point and susceptibility at $K^{\chi'}_{\max}$, should scale for sufficiently large systems as

$$< |m| > |_{\inf} \propto L^{-\beta/\nu} \tag{9}$$

$$\chi'_{\max}(L) = \chi'(K^{\chi'}_{\max}(L), L) = AL^{\gamma/\nu}, \tag{10}$$

where $L = \sqrt{N}$ defines the linear size of the lattice. As shown in Figure 7, the least-squares straight lines that fit our simulation data give $\beta/\nu = 0.132 \pm 0.007$ and 0.126 ± 0.008 for $\alpha = 0.5$ and 1.0, respectively. These values are in good agreement with those obtained for the Ising model on the Voronoi lattice (with constant J) [4]. However, while $\gamma/\nu = 1.75$ for the Ising model, the best fits to our susceptibility data (Fig. 8) indicate that $\gamma/\nu = 1.20 \pm 0.07$ and 1.45 ± 0.05 for $\alpha = 0.5$ and 1.0, respectively. More calculations with larger lattices and a large number of realizations are certainly necessary to verify if the universality class is the same or not.

Recent studies [3] have already focused on models that undergo second-order transitions in the pure system and apparently display Ising critical behavior when impurities are present. It is also interesting to note that systems with quenched randomness (site or bond) appear to be in the same universality class of the pure 2D Ising model. However, in most of these studies, the results do not seem to be sufficiently accurate to clarify this aspect.

In the present work, we have shown that, by considering the eight-states ferromagnetic Potts model on a Voronoi lattice with the coupling constant depending exponentially on the distance between neighbor sites, the order of the transition changes from first to second order. This is in agreement with the fact that randomness can change the order of the transition [3]. In addition, we have calculated the exponent ratios β/ν and γ/ν for the magnetization and susceptibility, respectively. For β/ν, the system appears to belong to the same universality class of the two-dimensional Ising model. On the other hand, the estimated values we found for the critical exponent γ/ν at two different values of $\alpha > 0.0$ differs significantly from each other and, moreover, from the Ising exponent reported in literature, $\gamma/\nu = 1.75$.

Our current work is devoted to the numerical calculation of a precise estimate for α_c, *i.e.*, the particular value of α where the transition changes from first to second order. In addition, we are performing extensive simulations with large lattice sizes and number of realizations to characterize the universality class of the present model.

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